

A little scholium on Hilbert-Rohn via the total reality of M -curves: Riemann's flirt with Miss Ragsdale

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Poincaré hat zuerst die Frage nach dem Gesamtverlauf der reellen Lösungen von Differentialgleichungen mit topologischen Mitteln behandelt.
Hellmuth Kneser, 1921, in *Kurvenscharen auf geschlossenen Flächen* [20].

ABSTRACT. This note presents an elementary proof of Hilbert's 1891 Ansatz of nesting for M -sextics, along the line of Riemann's Nachlass 1857 and a simple Harnack-style argument (1876). Our proof seems to have escaped the attention of Hilbert (and all subsequent workers) [but alas turned out to contain a severe gap, cf. Introduction for more!]. It uses a bit Poincaré's index formula (1881/85). The method applies as well to prohibit Rohn's scheme $\frac{10}{1}$, and therefore all obstructions of Hilbert's 16th in degree $m = 6$ can be explained via the method of total reality. (The same ubiquity of the method is conjectured in all degrees, and then suspected to offer new insights.) More factually, a very simple and robust phenomenon of total reality on M -curves of even order is described (the odd-order case being already settled in Gabard 2013), and it is speculated that this could be used as an attack upon the (still open) Ragsdale conjecture for M -curves (positing that $|\chi| \leq k^2$). Of course a giant gap still remains to be bridged in case the latter conjecture is true at all. Alas, the writer has little experimental evidence for the truth of the conjecture, and the game can be a hazardous one. However we suspect that the method of total reality should at least be capable of recovering the weaker Petrovskii bound, or strengthened variants due to Arnold 1971. This text has therefore merely didactic character and offers no revolutionary results, but tries to reactivate a very ancient method (due basically to Riemann 1857) whose swing seems to have been somewhat underestimated, at least outside of the conformal-mapping community.

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1 Introduction

[20.04.13] The flirt suggested in our title is a fictional one, which cannot have occurred between Bernhard Riemann (1826–1866) and Virginia Ragsdale (1870–1945). However when the latter came to visit Klein and Hilbert in Göttingen (ca. 1903?) the spirit of Riemann was most vivid than ever and we shall try to speculate about a direct connection between the works of both scientists.

In a previous paper (Gabard 2013 [5]), we made an essay to connect a certain theory of *total reality* rooted in Riemann’s work on conformal representation with Hilbert’s 16th on real algebraic curves by extrapolating a bit (hopefully not fallaciously) the eclectic visions of V. A. Rohlin 1978. We do not repeat here the vast body of knowledge and array of conjectures accumulated in both disciplines and hope to have made there sufficiently explicit a possible deep interpenetration of both topics. The aim of this note is to illustrate the method of total reality on a more concrete terrain, namely *Hilbert’s nesting Ansatz* for M -sextics which is undeniably the first nontrivial result (1891) paving the way toward the general formulation (in 1900) of Hilbert’s 16th (isotopic classification of real plane algebraic curves, i.e. how ovals of such curves are distributed among themselves, nested and mutually positioned).

Bibliographical references.—To keep the bibliography of the present text within reasonable limits, whenever a work is cited by specifying only its author name and date (of publication) we refer the interested reader to the extensive bibliography compiled in Gabard 2013 [5].

Glossary of synonyms.— • Harnack(-maximal) curve= M -curve, jargon of Petrovskii 1938, where M stands probably for maximal.

★★★ *Very Important Warning (Mea Culpa) (added in proof the [22.04.13]).*— After having posted this note on the arXiv (yet the day prior to its diffusion), we noticed that our proofs of the Hilbert and Rohn theorems (2.1) and (2.2)—via the method of total reality due to Riemann—contains a serious gap. *Exercise:* detect our mistake without reading the next hint in tiny calligraphy.

Hint: it seems that we have overlooked the possible presences of centers singularities (infinitesimally like concentric circles) in the foliation (also contributing to positive indices). Such centers may occur when curves of the pencil of quartics contract an oval toward a solitary node.

If optimistic, this defect can perhaps be repaired, admittedly after much more efforts. We decided to still publish this note for two reasons. First, in the hope that someone is able to arrange a proof of Hilbert (and Rohn) along the method of Riemann. Second, our main result (2.3) on the total reality of M -curves is not affected by this issue and complements the odd-degree case settled in Gabard 2013 [5, Thm 31.12, p.402]. Alas, this main result has very basic character and should merely be regarded as a first step toward deeper problems à la Hilbert, Rohn or Ragsdale (that we are presently unable to tackle). Therefore, it is evident that both our title (and abstract) are much immature (not to say pathetic), but we left them unchanged deliberately in the hope to attract more qualified workers to the question. Of course anybody able to complete the programme from Riemann-to-Hilbert can build upon our free-source file in case its historical aspects seem of some didactic value.

1.1 Hilbert’s Ansatz: overview of all known proofs (Hilbert 1891/1900/01, Wright 1907, Kahn 1909, Löbenstein 1910, Rohn 1911/13, Donald 1927, Hilton 1936, Petrovskii 1933/38, Kervaire-Milnor 1961, Arnold 1971, Rohlin 1974/78)

• *Hilbert turning to a geometer.*—In 1891, in a genius stroke without any antecedents, Hilbert advanced (without proof, quite uncharacteristic of his style) the conclusion that a sextic curve which is Harnack-maximal (i.e. with the maximum number 11 of ovals¹) cannot have all its ovals *unnnested* lying outside each

¹As pointed by Elias Boulé it seems that 11 is also the number of planets circulating in the Solar system, when nano-objects like Ceres are included into the count. Ceres is the greatest,

other. Hilbert 1891 [11] confessed in a footnote his proof to be exceptionally complicated and highbrow, more precisely:

“Diesen Fall $n = 6$ habe ich einer weiteren eingehenden Untersuchung unterworfen, wobei ich — freilich auf einem außerordentlich umständlichen Wege — fand, daß die elf Züge einer Kurve 6-ter Ordnung keinesfalls sämtlich außerhalb un voneinander getrennt verlaufen können. Dieses Resultat erscheint mir deshalb von Interesse, weil er zeigt, daß für Kurven mit der Maximalzahl von Zügen der topologisch einfachste Fall nicht immer möglich ist.”

It required several generations of workers until the method of Hilbert reached full maturity. The detailed story is probably best recorded in Gudkov’s survey of 1974 [7], but let us sketch it briefly (while adding some “inedited” items to the narration). Hilbert himself seems to have been quite fluctuant in evaluating the level of rigor of his proof. As far as we know, he never published himself a proof, but supervised two Göttingen Dissertations on the question (Kahn 1909 [14] and Löbenstein 1910 [23]), which apparently turned out to be inconclusive. At least this is the opinion of both Rohn 1913 [31, p.178] and Gudkov 1974 (p.41), who actually asserts that on their own admissions those writers (Hilbert’s girls) confessed to have failed proving nonexistence of a C_6 of unnested type 11. Yet, it is slightly puzzling that Hilbert 1909 [13] qualified the proof of Kahn-Löbenstein as complete, more precisely:

“[...] eine ebene Kurve 6-ter Ordnung hervorgehen, die aus elf außerhalb voneinander getrennt verlaufenden Zügen bestände. Daß aber eine solche Kurve nicht existiert, ist einer der tiefstliegenden Sätze aus der Topologie der ebenen algebraischen Kurven; derselbe ist kürzlich von G. Kahn und K. Loebenstein (Vgl. die Göttinger Dissertationen derselben Verfasserinnen.) auf einem von mir angegebenen Wege bewiesen worden.”

- *Rohn*.—Then came Rohn 1911/13 who elaborated Hilbert’s method in much more details. Yet according to Academician D. A. Gudkov (still 1974) this was still not rigorous enough and required some consideration of dynamical system à la Andronov-Pontryagin (*systèmes grossiers*, alias structural stability) to become logically robust. The method was then christened the *Hilbert-Rohn method*.

What came next? As reported in Gudkov 1974 (p.42), a *completely non-rigorous, descriptive attempt* of proof (of Hilbert’s Ansatz) was made in Donald 1927 (repeating apparently the earlier inconclusive attempt of Wright 1907), all expressing the same methodology as Hilbert. H. Hilton 1936 devoted a paper to a criticism of Donald’s article.

- The real breakthrough occurs with Petrovskii 1933/38 who supplies a universal inequality valid in all degrees pinching (from both sides) the Euler characteristic χ of the Ragsdale membrane bounding the ovals from inside. One side of Petrovskii’s inequalities reads $\chi \leq \frac{3}{2}k(k-1) + 1$, where $k := m/2$ is the semi-degree of the curve of even order $m = 2k$. This implies Hilbert’s Ansatz, and of course much more. His proof is an explosive cocktail: Euler-Jacobi-Kronecker interpolation formula combined with Morse theory (1925). For sextics ($m = 6$, hence $k = 3$), Petrovskii’s upper-bound is 10 and so the curve with 11 unnested ovals is ruled out (its χ being 11). Hilbert’s Ansatz is (re)proved, or even proved *for the first time*, if we accept Gudkov’s (sibylline) critiques to both Hilbert and his students as well as toward Rohn. For this and other achievements, Petrovskii is often regarded by Arnold as one of the deepest 20th-century scholar of all Russia.

- Another proof (and perhaps the next one historiographically) is due (or rather follows) from Kervaire-Milnor 1961 [17]. In there concise PNAS-note, this eminent tandem proves what later (or former?) went known as the *Thom conjecture* in the special case of homology classes of degree 3. The Thom conjecture is the assertion that a smooth oriented surface in the complex projective plane $\mathbb{C}P^2$ (the 4-manifold of all unordered pairs of points on the 2-sphere) has genus at least as big as that of an algebraic curve of the same degree, namely $g = \frac{(m-1)(m-2)}{2}$. This conjecture of Thom went validated by Kronheimer-Mrowka in 1994, but its degree 3 case is much older (1961 as we just said) and incidentally much based upon work of the superhero V. A. Rohlin (ca. 1951).

and first detected (1801), asteroid gravitating somewhere between Mars and Jupiter with a diameter of about 1000 km.

Now suppose given a sextic with 11 unnested ovals. Since it is Harnack-maximal the ovals disconnect the complexification (by Riemann’s definition of the genus). This is a remark of Klein 1876, which naively amounts to visualize the Galois symmetry of complex conjugation as a reflecting mirror about a plane leaving invariant a pretzel of genus g symmetrically sculpted in 3-space and cutting the plane along $g + 1$ ovals (cf. Fig. 1a for the case $g = 3$). Dissecting one half of the curve gives a bordered surface which pasted with the ovals-insides creates a surface of genus 0 (topological sphere) whose degree (in the homological sense) is of course the halved degree of the sextic, namely 3. Rounding corners (if necessary?) gives a smooth surface whose degree is 3 but of genus 0 only, hence beating that of a smooth cubic of genus 1. Thom’s conjecture (i.e. Kervaire-Milnor’s theorem) is violated and Hilbert’s Ansatz proved (via pure topology).

- *Arnold-Rohlin’s era.*—The story does not finish here, and other spectacular simplifications of Hilbert’s Ansatz came under the pen of V.I. Arnold 1971, and his companion V. A. Rohlin 1974. Arnold 1971 established the congruence $\chi \equiv k^2 \pmod{4}$ (valid actually for all dividing curves, not only M -curves). This prohibits the “Hilbert sextic” with 11 unnested ovals. In 1974 Rohlin found *Rohlin’s formula* $2(\pi - \eta) = r - k^2$, where r is the number of ovals while π, η are resp. the number of positive and negative pairs of ovals (defined by comparing orientations induced by the complexification of a dividing curve with those coming from the bounding annulus for the nested pair of ovals). This formula implies formally Arnold’s congruence (compare e.g., Gabard 2013 [5, p. 258, Lemma 26.11]), and also implies Hilbert’s Ansatz. Indeed in the absence of nesting, the left-side of Rohlin’s formula vanishes and so $r = k^2 = 3^2 = 9$, which is not equal to 11. So Rohlin’s formula is the dancing queen of what can be done in the most elementary way. Its proof involves capping off the 2 halves of the dividing curve by the bounding discs of all ovals (hence overlapping violently in case of much nesting), as to construct two singular 2-cycles in $\mathbb{C}P^2$ whose intersection is computed after pushing both objects in general position.

At this stage nobody cared anymore to prove Hilbert’s Ansatz as the (Arnold-Rohlin) proof was nearly “*from the Book*”. Is the reader at this stage convinced of the truth of Hilbert’s Ansatz just on the basis of what is to be found in our note? Presumably not as we did not presented any self-contained proof, but this state of affairs will be remedied in the sequel of this text.

What came next? Probably several details but the level of perfection of Arnold-Rohlin (with slight improvements by Wilson 1978) was so drastic that it left little room for any further imagination.

- In Jan. 2013, we discovered another little explanation of Hilbert’s Ansatz. Suppose the sextic curve to have 11 unnested ovals. It seems a reasonable folly to expect that empty ovals of curves can always be contracted to points (solitary nodes) via a continuous deformation of the coefficients (and this, despite the rigidity reputation of algebraic objects). An oval here is said to be *empty* if looking inside of it, one sees no other smaller ovals. Such principles of contractions were actually exploited by Klein in 1892 (if not earlier) and also form the content of a conjecture of Itenberg-Viro 1994, which posits that *any* empty oval of an algebraic curve can be shrunk to a solitary node. This is a truly remarkable conjecture which has neither been proved nor been refuted up to present days.

Let us, cavalier, assume a stronger version of this conjecture stipulating that *all* empty ovals can be shrunk *simultaneously* (synchronized death of all empty ovals). Apply this contraction to an unnested M -sextic with 11 unnested ovals to see its underlying Riemann surface of genus 10 strangulated into 2 pieces of degree 3 intersecting in 11 points (Fig. 1a). But Étienne Bézout told us a long time ago that 2 cubics intersect in $3 \cdot 3 = 9$ points. Hilbert’s nesting Ansatz is proved modulo the (unproven) contraction principle. Alas, the writer does not know if the collective contraction principle just employed holds true in degree 6, but this could be quite likely as the shrinking of *any* single oval is a result of Itenberg 1994, based on the marvellous technology of Nikulin 1979 (K3 surfaces,

global Torelli for them, etc.)

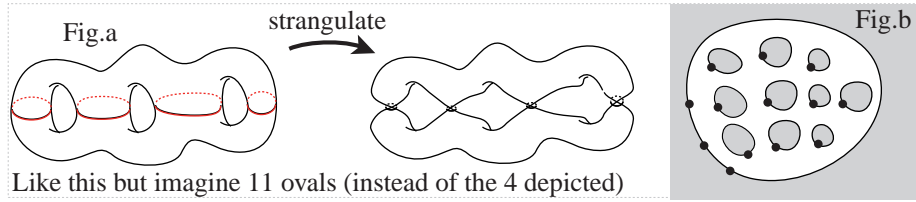


Figure 1: Contracting all the ovals of an unnested curve toward solitary nodes with imaginary-conjugate tangents

It may be wondered if our strangulation proof (as heuristic as it is) was known to Hilbert (or even Klein). It should be remembered that principles of contraction were often used by Klein (say from 1876 up to 1892) and so it is quite likely that those Göttingen scholars may have thought about this method at least as supplying some heuristic evidence. (Alas, we know about no trace left in print.)

- Yesterday evening [19.04.13], we found another pleasant argument based on the method of total reality. This method has historical origins in the theory of conformal mappings, especially Riemann's Nachlass of 1857. Many subsequent workers were involved in this theorem of Riemann, and we merely cite them in cascade referring again to Gabard 2013 [5] for exact references: Schottky 1875/77, Wirtinger ca. 1900 (unpublished), Enriques-Chisini 1915, Bieberbach 1925, Grunsky 1937, Courant 1939, Wirtinger 1942 (published this time), Ahlfors 1947/50, A. Mori 1951, . . . , Huisman 2001, and many others in between.

If this Riemann Nachlass is interpreted extrinsically along the method used by Harnack 1876 (to prove the after-him called bound $r \leq g + 1$ on the number r of ovals), we get a very simple derivation of Hilbert's Ansatz as we shall explain in the next Sec. 2.1. This is the (modest) goal of this note, but we strongly suspect that when applied more cleverly Riemann's Nachlass could crack the Ragsdale conjecture or at least affords simple proofs of the myriad of estimates due to Petrovskii or his (greatest admirer) Arnold. So our game is an attempt to shrink back everything to Riemann via the method of total reality.

As far as we are concerned, we have to acknowledge some pivotal inspiration from the paper by Le Touzé 2013 [4], where the total reality of quintics is explained in a synthetic fashion (i.e. a Harnack-style argument with boni-intersections gained by topology or algebra). A simple extension thereof to all curves of *odd degrees* is given in Gabard 2013 [5, Thm 31.12, p.402]. In that work we failed to treat the case of even order curves and this is remedied below (Sec. 2.3) by showing that total reality is likewise a very simple matter (using the parity of intersection between ovals).

2 Proofs

2.1 A 2 seconds proof of Hilbert's Ansatz

Theorem 2.1 (Hilbert 1891).—*A real sextic curve cannot have 11 unnested ovals.*

Proof. Inspired by the method of total reality, we consider a certain ancillary pencil of quartics. Writing down monomials along increasing degrees

$$1, \underbrace{x, y}_2, \underbrace{x^2, xy, y^2}_3, \underbrace{x^3, x^2y, xy^2, y^3}_4, \text{etc.}$$

(best visualized as a pyramid à la Newton) we see that quartic curves depend upon $1 + 2 + 3 + 4 + 5 = \frac{6 \cdot 5}{2} = 15$ parameters (the coefficients). Since we are only interested in the equation up to homothety there are only 14 essential parameters, and so by linear algebra there can be assigned 13 basepoints to a pencil of quartics.

Consider now the 11 ovals of the curve C_6 as pigeonholes where to range the 13 basepoints. Distribute them injectively among the 11 ovals while placing the 2 remaining ones on the same oval (compare Fig. 2b if necessary). By a principle due to Möbius-von Staudt (and massively used by Zeuthen 1874 and Harnack 1876) we know that 2 ovals in $\mathbb{R}P^2$ intersect always in an even number of points (counted by multiplicity if necessary). Accordingly any curve C_4 of our pencil of quartics has one boni-intersection on each oval since we are always imposing an odd number of basepoints on each oval. So $13 + 11 = 24 = 4 \cdot 6$ real intersections are granted by intersection theory à la Möbius-von Staudt. This is the maximum permissible by Bézout. We speak of a phenomenon of *total reality*.

Consider next the (mildly singular) foliation induced by this pencil of quartics on the inside R of the 11 ovals, which may be seen as a special case of the *Ragsdale membrane* bounding the ovals orientably “from inside”. Since there is no nesting this membrane R is merely a disjoint union of 11 (topological) discs. It is convenient to double this membrane to get $2R$, a union of 11 spheres.

Now 13 basepoints are assigned on the boundary $\partial R = C_6$ but a pencil of quartics has 16 basepoints (Bézout once more). So there is 3 unassigned basepoints, on which we know very little. In the worst case those 3 points will land inside of the ovals. Otherwise they can land on the ovals, or eventually outside of them.

Apply Poincaré’s index formula 1885 (announced 1881) telling us that the sum of indices of a foliation² is equal to the Euler characteristic of the surface. Each foyer-type singularity (infinitesimally like the pencil of lines through a point) has an index of +1. In an algebraic foliation, basepoints induce such foyers and neglecting crudely all singularities of negative indices gives the estimate:

$$\chi(2R) = \sum \text{indices} \leq 13 + 3 \cdot 2 = 19,$$

since all the 3 unassigned basepoints contribute for at most 2 foyers (one on each “face” of the double) [Panoramix-double-fax], while the 13 comes of course by doubling the semi-foyers visible at each of the 13 basepoints assigned on the C_6 . On the other hand, $\chi(2R) = 2\chi(R) = 2 \cdot 11 = 22$. This is arithmetical nonsense and Hilbert’s theorem is proved. ■

Historiography.—Our proof uses *Poincaré’s index formula* (1881/85), of course very well-known to Hilbert (cf. e.g. the citation by Hellmuth Kneser, one of Hilbert’s student, on the front-page of this note). The (pre)history of Poincaré’s formula is probably best recorded in von Dyck 1888, where (vague) forerunners are listed like Gauss 1839, or Kronecker 1869, and many others.

2.2 Rohn’s prohibition of the scheme $\frac{10}{1}$ via total reality

[21.04.13]. Applying the above argument to the dual *non-orientable* membrane, say N , bounding the ovals from outside proves Rohn’s prohibition (1913 [31]) of the scheme $\frac{10}{1}$ where 10 ovals are enveloped in a larger oval.

Theorem 2.2 (Rohn 1913).—*An M -sextic curve C_6 cannot have 10 ovals enveloped in a larger eleventh oval.*

Proof. As above we consider a total pencil of quartics with 13 basepoints distributed injectively on the 11 ovals safe that one of the oval absorbs 3 basepoints. Consider the algebraic foliation induced on the anti-Ragsdale membrane N discussed above (compare Fig. 1b).

Applying Poincaré’s index formula to the doubled membrane $2N$, we find

$$\chi(2N) = \sum \text{indices} \leq 13 + 3 \cdot 2 = 19.$$

²Poincaré 1885 worked this case too, as opposed to simply flows which are orientable foliations by Kerékjártó-Whitney 1925/1933.

On the other hand $\chi(2N) = 2\chi(N)$, and N is the union of a Möbius band (with $\chi = 0$) plus 10 replicas of the 2-sphere S^2 with $\chi = 2$, whence $\chi(N) = 0+10 = 10$ and therefore $\chi(2N) = 20$. The proof is complete. ■

It is puzzling that those arguments escaped Hilbert and Rohn. Philosophically it seems that the cause is that those workers were too much algebraically inclined as opposed to the pure geometry of Riemann and Poincaré. So our argument represents a little victory of (the angel of) geometry over (the devil of) algebra, as would say H. Weyl.

More seriously, the 2 proofs given above are fundamental in completing the programme sketched in the Introd. of Gabard 2013 [5]. There we explained how the Rohlin-Le Touzé phenomenon of total reality explains *nearly all* prohibitions of Gudkov's census solving Hilbert's 16th in degree $m = 6$. The “*nearly all*” referred precisely to the fact that this missed the 2 schemes 11 and $\frac{10}{1}$ prohibited by Hilbert and Rohn respectively. Since we are now also able to treat those cases via total reality, we see that in degree $m = 6$ the method of total reality is ubiquitous and universal. Of course we conjecture this to be a general issue for all m , compare again the Introd. of Gabard 2013 [5].

2.3 Total reality of M -curves of even order (the punched card device of Harnack-Le Touzé-Gabard)

[17.04.13] Our former work (Gabard 2013 [5]) failed to assess total reality of M -curves of even degree in the strong sense of knowing where to assign basepoints. (For the weak sense reminiscing perhaps the *Brill-Noetherschen Restsatz*, see [5, Thm 31.8, p.399].) Now we show that a very simple device (already used in Harnack 1876) grants (strong) total reality of M -curves in the even degree case too. (For the odd degree case see [5, Thm 31.12, p.402]).

It would cause no trouble to write down directly the general result and proof but this not the way one usually discovers truth, so let us work more peacefully. (The pressed reader can directly move to (2.3).)

Let us start with degree $m = 6$ (sextics). Here we look (in accordance with the general theorem à la Brill-Noether ([5, Thm 31.8, p.399]) or just Riemann-Bieberbach) to curves of degree $m - 2 = 4$, i.e. quartics. Those may be assigned to visit $B = \binom{4+2}{2} - 2 = 13$ basepoints while still moving in a pencil. On the other hand our M -sextic has $M = 11$ ovals. How to distribute basepoints as to ensure total reality of the quartics-pencil. A priori we may distribute the 13 basepoints on the 11 ovals (surjectively and in very random fashion), but then only $22 < 4 \cdot 6 = 24$ real intersections are granted. Let us be more specific. Suppose that we distribute injectively 11 basepoints on the 11 ovals, while placing the 2 remaining points on 2 distinct ovals (cf. the black dots on Fig. 2a). Then a C_4 of the pencil has $2 \cdot 2 + 9 \cdot 2 = 22$ real intersections granted (compare again Fig. 2a, where the white dots are extra intersections gained for parity reasons of the intersection of two ovals=even degree circuits in older jargon). This is not enough for total reality to be valid at $24 = 4 \cdot 6$. If however our “punching-card machine” assigns the 2 additional points on the same oval (like on Fig. 2b), then we get $1 \cdot 4 + 10 \cdot 2 = 24$ real intersections and total reality is demonstrated (compare again Fig. 2b counting now also the bonus intersections materialized by white bullets). Again, we used the classical fact that 2 ovals have an even number of intersections counted by multiplicity.



Figure 2: Repartition of basepoints in black, and extra boni-intersections gained by the punching-card trick in white

Optionally, it is a pleasant exercise to check that the same trick and (little)

miracle of boni works in degree 8. Now since we expect a pencil of sextics we have $B = \binom{8}{2} - 2 = 26$ basepoints available, while we have Harnack's bound many, i.e., $M_8 = g + 1 = \frac{7 \cdot 6}{2} + 1 = 22$, ovals at disposal. Again to create as much boni intersections as possible, it is valuable to disperse the 4 excess/surplus basepoints as 2 groups of height 2 (Fig. c). Then by the evenness principle for intersecting ovals we have $2 \cdot 4 + 20 \cdot 2 = 48 = 6 \cdot 8$ real intersections granted (the maximum permissible by Bézout). Total reality is proved. It may be noted that choosing a distribution like Fig. d, where the 4 extra *bases/eyes* (=abridged for basepoints) are concentrated on a single oval, total reality is likewise granted (as $1 \cdot 6 + 21 \cdot 2 = 48 = 6 \cdot 8$). More basically, without arithmetics, we may infer this by noticing that as the degree is odd in restriction to each pigeonhole materializing an oval, we gain also one white-bullet above all pigeonholes as in the former case, whence total reality.

As customary in such games, it is straightforward to extend to any degrees and we arrive at the following result (in philosophical substance known to Harnack 1876, or Enriques-Chisini 1915, or Le Touzé 2013, and of course many others like Joe Harris, Johan Huisman, etc.):

Theorem 2.3 *Given any M -curve of even degree m , the pencil of curves of order $m - 2$ assigned to visit a repartition of basepoints having odd “degree” on each oval is totally real. Further, and in accordance with the Riemann-Schottky-Enriques-Bieberbach theorem, the pencil possesses exactly one mobile point circulating along each real circuit.*

Proof. The number of basepoints for a pencil of $(m - 2)$ -tics is $B = \binom{(m-2)+2}{2} - 2 = \binom{m}{2} - 2$. Harnack's bound for the given m -tics C_m is $M = \binom{m-1}{2} + 1$. Hence the excess of basepoints over the number of ovals is $B - M = [1 + 2 + \dots + (m - 1)] - 2 - [1 + 2 + \dots + (m - 2)] - 1 = m - 4$.

We may for instance share out those $m - 4$ extra basepoints as on Fig. 2e, i.e. by splitting them in $\frac{m-4}{2}$ groups of “height” 2. This is arithmetically meaningful as m is supposed even. Then, on counting real intersections forced by intersection theory (of ovals), we are granted of (cf. again Fig. e) $4(\frac{m-4}{2}) + 2[M - \frac{m-4}{2}] = 2(m-4) + 2M - (m-4) = 2M + (m-4) = 2[\frac{(m-1)(m-2)}{2} + 1] + (m-4) = (m-1)(m-2) + (m-2) = m(m-2)$, and total reality is demonstrated.

Of course as one bonus intersection is gained on each oval the count works whenever the repartition has odd degree on each oval and the asserted total reality is established.

The last clause of the statement (analogy with Riemann *et al.*) follows by noticing that the boni intersections (white bullets on Fig. 2e) are unique on each oval. ■

To what is this (theorem) useful at all? Always keep in mind that we are geometers not (so much) obnubilated by the magics of arithmetics. What could be desired is an intelligence capable of visualizing such pencils, and playing maybe with the Poincaré-von Dyck index formula (1885/88 respectively). (Recall that Ragsdale 1906 cites von Dyck 1888 precisely for this purpose, yet not so surprising as both are docile students of Klein.) Granting this visualization (or just an arithmetical/combinatorial corollary of it like Poincaré's index formula) one could maybe infer a new proof of Hilbert's prohibition of the (unnested) scheme 11. [*Added in proof* [20.04.13]: This is indeed possible see (2.1).]

Even more ambitiously, we could dream that a thorough inspection of such pencils (maybe combined with the Arnold/Rohlin tricks of splitting and closing Klein's orthosymmetric half by suitable limbs of real membranes) could imply a proof of the elusive *Ragsdale conjecture* $\chi \leq k^2$. The so-called *Arnold surface* (i.e. Klein's half glued with Ragsdale's membrane) is always embedded but alas not ever orientable (else Ragsdale would be a trivial consequence of Thom, cf. [5, Sec. 33]). By-standing to the former there is also (what we propose to call) *Rohlin's surface* which has the advantage of orientableness, but a “singular chain” (in the jargon of Lefschetz-Alexander-Eilenberg) which is not embedded in general. For the definition of this surface it suffices us to say that it is

the one involved in the proof of Rohlin's formula. One could try to inspect the intersection of (say) an imaginary member of the pencil with the semi-Riemann surfaces of Arnold or better Rohlin (which is orientable hence defining an integral homology class). Further, imaginarieness amounts to unilaterality (in the sense of Gabard 2006) under total reality: this is just to say that all intersections have to be located in the same half (whence our pompous name "unilaterality") apart from those coming from the assigned real basepoints. We admit that this approach to Ragsdale is probably overoptimistic, yet it seems wise to leave open any possible strategy toward the elusive conjecture. For another surely much more mature attack, cf. Fiedler's programme sketched in his terrible letter dated [14.03.13] reproduced below.

••• [14.04.13] dimanche 14 avril 2013 10:33:08, Fiedler wrote:

Dear Alexandre,

just to let you know some old results, which could perhaps be worth to be explored in more generality.

In the mid 90' I have tried to prove the Ragsdale conjecture for M -curves with an idea coming from knot theory: bring the object first to its most symmetric position.

DEFINITION: A real curve X is said to be symmetric if it is invariant under a (non-trivial) holomorphic involution s of the complex projective plan.

The idea was to deform first the curve into a symmetric one and then to explore the additional information coming from pencils of lines which are real simultaneously for both real structures on X , namely $conj$ and $s.conj = conj.s$.

It has failed miserably, because I have proven the following theorem.

THEOREM 1. For a symmetric M -curve of degree $2k$ the following refinement of the Gudkov-Rokhlin congruence holds:

$$p - n = k^2 \bmod 16.$$

So, roughly half of the M -curves are not symmetric. It is amazing that some conjectures are false in general but true for symmetric M -curves. Hilbert's conjecture: because the Gudkov curve is not symmetric. Viro's conjecture: because an M -curve of degree 8 with a nest of depth 3, which has an odd number of innermost ovals, can not be symmetric. On the other hand the idea was not so bad because I have proven the following theorem.

THEOREM 2. If a symmetric M -curve of degree $2k$ has a nest of depth $k - 2$ then $p - n \leq k^2$ and if equality holds then the Arnold surface A^+ is orientable.

These results were never published, because I was already too much into knot theory. But you can find some information about it in a paper of Erwan Brugalle and in a paper of my student Sebastien Trilles. Very best, Thomas

3 Speculations

3.1 Flirting with Miss Ragsdale

[19.04.13] As we noticed in the previous section there is some chance that total reality can crack the still open (and elusive) Ragsdale conjecture for M -curves of even degrees, namely the estimate $\chi \leq k^2$. (NB: the full-Ragsdale conjecture posits $|\chi| \leq k^2$.)

Our idea is based on the previous total reality phenomenon (2.3) for plane M -curves of even order $m = 2k$. This phenomenon is merely a Harnack-style argument with boni-intersections gained by evenness of the intersection of 2 ovals, yet it looks so robust and easy that it seems a reasonable attack upon Ragsdale. To add some spiciness to our strategy it should always be remembered that total reality truly belongs to Ahlfors 1950 (or maybe Klein according to Teichmüller 1941), yet on the case at hand of M -curves it is really due to Riemann's Nachlass of 1857. This being said we expect a big flirt between Riemann 1857 and Miss Ragsdale 1906.

Now the idea would be that some intelligence able to visualize properly this pencil (while extracting the relevant combinatorial aspects) should be able to derive from the total reality of such pencils the estimate $\chi \leq k^2$ (and perhaps its opposite $-k^2 \leq \chi$ too). As we said the trick could be to intersect an imaginary (hence unilateral) member of the pencil with the *Rohlin surface* obtained by aggregating the bounding discs of all ovals. The little technical difficulty is that one requires to understand the intersection indices so obtained. This game still escapes me slightly but is well understood by Arnold and Rohlin, compare e.g. the proof of Rohlin's formula. One of the additional difficulty is that the pencil of $(m-2)$ -tics will have non-assigned basepoints and those create additional intersections somewhat harder to control since their location is not known a priori (in contrast to the assigned basepoints). We hope to discuss this issue in more detail later.

Perhaps another general philosophical comment. As we emphasized the method of total reality used in (2.3) for M -curves of even order is just an avatar of a Harnack-style argument. Historically it may also be remembered that the prototype for this sort of reasonings goes back to Zeuthen 1874, who impressed much Klein and so indirectly Harnack. Of course Zeuthen himself refers back to Möbius and von Staudt who expressed in modern vocabulary fixed what we call nowadays the intersection theory of $\mathbb{R}P^2$. So our strategy toward Ragsdale bears some close analogy with Harnack's synthetic proof of the so-called Harnack inequality $r \leq g + 1$. Hence if Ragsdale estimate $\chi \leq k^2$ (or its general version with absolute value $|\chi| \leq k^2$) is correct, it is likely that its proof proceeds along a similar line than that of Harnack's inequality which is so-to-speak the most fundamental estimate for the topology of real curves. Maybe this vague analogy gives another weak evidence that we are on the right track toward proving Ragsdale [or related results à la Petrovskii-Arnold].

A last philosophical remark is in order. As we all know Klein 1876 offered a somewhat more conceptual (or intrinsic) justification of $r \leq g + 1$ by using merely topology, as opposed to the synthetic geometry of Harnack. As a rule Klein's argument is conceptually somewhat more limpid than Harnack's which is a bit tricky arithmetics/combinatorics. Accordingly one could also suspect a topological proof of Ragsdale somewhat easier than via total pencils. In substance this could be our crude but erroneous approach via Thom's genus (lower) bound ([5, Sec. 33]) or the programme sketched by Fiedler using knot theory. Notwithstanding we may expect that our synthetical strategy has still some good chance to crack Ragsdale, and we hope being able to attack this question in the future.

3.2 A disappointing estimate with zero-information on the unassigned basepoints

[19.04.13, but TeXified 21.04.13] If we ape directly the proof of Theorem 2.1, i.e. Hilbert's Ansatz via Riemann's Nachlass and Poincaré's index formula to the case of a general even degree $m = 2k$ M -curve, we get an estimate which is extremely disappointing when $m > 6$. This will be exposed right below, and the dream should be to get a sharper estimate by trying to control better the location of the unassigned basepoints.

Let C_m be a plane M -curve of even degree $m = 2k$. By Theorem 2.3 we have a phenomenon of total reality for a pencil of curves of degree $(m-2)$ assigned to visit $B = M + (m-4)$ basepoints (cf. 1st paragraph of its proof).

So applying Poincaré's index formula to the (doubled) Ragsdale membrane, $2R$, we get the following bound after noting that there are $(m-2)^2 - B$ unassigned basepoints:

$$\begin{aligned} \chi(2R) &= \sum \text{indices} \leq B + 2[(m-2)^2 - B] \\ &= 2(m-2)^2 - B \\ &= 2(m-2)^2 - M - (m-4) \end{aligned}$$

$$\begin{aligned}
&= 2(m-2)^2 - \frac{(m-1)(m-2)}{2} - 1 - (m-4) \\
&= 2(2k-2)^2 - (2k-1)(k-1) - 1 - (2k-4) \\
&= 8(k-1)^2 - (2k-1)(k-1) - 1 - 2(k-2) \\
&= 8(k-1)^2 - (2k-1)(k-1) - 2(k-1) + 1 \\
&= (k-1)[8(k-1) - (2k-1) - 2] + 1 \\
&= (k-1)[6k-9] + 1 \\
&= [(k-1)3(2k-3)] + 1.
\end{aligned}$$

Therefore

$$\chi = \chi(R) \leq \frac{3}{2}(k-1)(2k-3) + \frac{1}{2}.$$

While this is interesting for $m = 6$ (as we saw), for $m = 8$ (so $k = 4$) this bound is useless, yielding only $\chi \leq \frac{3}{2}3 \cdot 5 + \frac{1}{2} = \frac{45}{2} + \frac{1}{2} = 23$, which is stupid as by Harnack we know $\chi \leq M = 22$. Of course Petrovskii bound is even much better yielding $\chi \leq \frac{3}{2}k(k-1) + 1 = \frac{3}{2}4 \cdot 3 + 1 = 19$. Further asymptotically our bound is $\approx 3k^2$ which is completely useless in comparison to Harnack's bound $\chi \leq M \approx 2k^2$

So we see that our method needs to be refined and there is of course much maneuvering room to do this, e.g. taking into account singularity of negatives indices and/or trying to predict the location of the unassigned basepoints. (It is perhaps here that deep predestination process of algebraic geometry à la Euler-Cayley-Bacharach or the Euler-Jacobi-Kronecker interpolation formula) have to enter into the scene. At this stage, it is safe to leave the topic to other more qualified workers (the dream being to crack Ragsdale's conjecture, and more modestly to reprove Petrovskii or the strengthened version due to Arnold!)

As a last loose idea it is important that the phenomenon of total reality always implies one point circulating on each circuit (=ovals). By virtue of the holomorphic character of the Riemann-Ahlfors map this gives raise to a dextrogyration, i.e. points moves compatibly with the complex orientation induced on the ovals (Rohlin's jargon). This and other dynamical principles should perhaps aid to predict the location of the unassigned base points.

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original claim, namely the dividing character is not deduced a priori from total reality (as Rohlin claimed being able to do), but rather the dividing character is taken as granted via the Rohlin-Kharlamov-Marin congruence while total reality of the pencil of cubics is built upon this preliminary knowledge. Hence it could still be of some interest to reconstruct a proof *purely a priori* assuming of course that there is a such. This looks quite likely, yet apparently quite elusive to implement.] ♡??

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- [10] P. Hartman, *Ordinary Differential Equations*, John Wiley, New York, 1964.
- [11] D. Hilbert, *Über die reellen Züge algebraischen Kurven*, Math. Ann. 38 (1891), 115–138; or Ges. Abhandl., Bd.II. [♠ where Hilbert’s 16th problem (Paris 1900) starts taking shape, in the sense of asking for the isotopy classification of plane smooth real sextics in $\mathbb{R}P^2 = \mathbb{P}^2(\mathbb{R})$ ♠ a method of oscillation is given permitting to exhibit a new scheme of M -sextic not available via Harnack’s method of 1876 (this is nowadays called Hilbert’s method) which is quite powerful (but not omnipotent) to analyze the topology of plane (real) sextics ♠ in particular Hilbert develops the intuition that a sextic cannot have 11 unnnested ovals (so must be nested), yielding some noteworthy form of complexity of algebraic varieties ♠ a complete proof of this assertion will have to wait for a longue durée series of attempt by his own students Kahn 1909 [14] Löbenstein 1910 [23] and especially Rohn 1911–13 [31]. All these attempts were judged unconvincing from the Russian rating agency (Gudkov) until the intervention of Petrovskii 1933–38 [29] and Gudkov 1948–1969, cf. e.g. Gudkov 1974/74 [7] ♠ p.418 (in Ges. Abh., Bd. II): “Diesen Fall $n = 6$ habe ich einer weiteren eingehenden Untersuchung unterworfen, wobei ich — freilich auf einem außerordentlich umständlichen Wege — fand, daß die elf Züge einer Kurve 6-ter Ordnung keinesfalls sämtlich außerhalb un voneinander getrennt verlaufen können. Dieses Resultat erscheint mir deshalb von Interesse, weil er zeigt, daß für Kurven mit der Maximalzahl von Zügen der topologisch einfachste Fall nicht immer möglich ist.” ♠ for the next episode in Hilbert’s pen, cf. Hilbert 1909 [13] where Hilbert ascribes to his students a complete proof of the result (inexistence of the unnnested scheme of 11 ovals)] ♡??
- [12] D. Hilbert, *Mathematische Probleme*, Arch. Math. Phys. (3) 1 (1901), 213–237; also in Ges. Abh., Bd. III. [♠ includes Hilbert’s 16th problem on the mutual disposition of ovals of plane curves (especially sextics), completely solved by Gudkov ca. 1969, cf. Gudkov-Utkin 1969] ♡??

- [13] D. Hilbert, *Über die Gestalt einer Fläche vierter Ordnung*, Gött. Nachr. (1909), 308–313; Ges. Abh. 2, 449–453. [♠ contains a good picture for the construction of Harnack-maximal sextic ♠ p.453, Hilbert ascribes to his students G. Kahn 1909 [14] and Löbenstein 1910 [23] a complete proof that a real sextic cannot have 11 unnested ovals (but that was not judged solid enough by subsequent workers, e.g. Rohn, Petrovskii, and Gudkov 1974 [7]): “[...] eine ebene Kurve 6-ter Ordnung hervorgehen, die aus elf außerhalb voneinander getrennt verlaufenden Zügen bestände. Daß aber eine solche Kurve nicht existiert, ist einer der tiefstliegenden Sätze aus der Topologie der ebenen algebraischen Kurven; derselbe ist kürzlich von G. Kahn und K. Loebenstein (Vgl. die Göttinger Dissertationen derselben Verfasserinnen.) auf einem von mir angegebenen Wege bewiesen worden.” ♠ nowadays there is five-minute proof of what Hilbert called one of the deepest problem in the topology of plane curves, via Rohlin’s formula ca. 1974–78, yet we believe that there is perhaps also a proof via the Ahlfors map (in the special case due to Riemann-Schottky-Bieberbach-Grunsky). This would be a fantastic project ★ *Upgrade* [20.04.13]—*Yes we can*, cf. Theorem 2.1 in this text] ♥??
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- [29] I. G. Petrovsky [Petrovskii], *On the topology of real plane algebraic curves*, Ann. of Math. (2) 39 (1938), 189–209. (in English of course.) [♠ where the jargon M -curve is coined, and where some obstruction is given (using the Euler-Jacobi interpolation formula), yielding perhaps (and according to Gudkov) the first proof, e.g. of the fact (first enunciated by Hilbert, Rohn, etc.) that a plane sextic cannot have 11 unnested ovals ♠ note however that Petrovskii himself validates Rohn’s proof of 1911 by writing on p. 189: “After a series of attempts the above mentioned theorem announced by Hilbert was at last proved in 1911 by K. Rohn(=Rohn 1911 [30]).” This contrast with Gudkov’s latter diagnostic (e.g. in Gudkov 1974 [7]) that even Rohn’s proof was not logically complete, though the method fruitful when suitably consolidated with Russian conceptions of roughness (Andronov-Pontryagin).] ♥??
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